SOME ASPECTS OF THE SOLUTIONS OF ROSSLER SYSTEM AND CHAOS

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Abstract. *In this paper we have studied the three dimensional non-linear system of differential equations by taking in consideration the two dimensional phase space of the Rossler system. The characteristics of the system are studied in reference to periodic, quasi-periodic and chaotic behavior. The conclusions are supported by means of diagrams of the trajectories obtained in two and three dimensional spaces, strange attractors and bifurcation diagrams.*

Keywords: *Nonlinear systems, trajectories, phase spaces, bifurcation diagrams, strange attractors.*

1. INTRODUCTION

A wide range of physical phenomena where there is a change in one quantity that occurs due to a change in one or more quantities can be mathematically modeled in terms of differential equations. Differential equations can be used to describe the motions of objects like satellites, water molecules in a stream, waves on strings and surfaces, etc. In this section we will take a review of some basic terminology associated with a system of differential equations.

1.1 System of Differential Equations [9]

Let $x_1(t),\ x_2(t), \dots, x_n$(t) be differentiable functions of a variable t defined on an interval I. Let $f_1,\ f_2, \dots, f_n$ be functions of $x_1,\ x_2, \dots, x_n$ and t. Then the set of n differential equations $X' = F(X,t)$, where $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$,

$X' = \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix}$ and $F = (f_1,\ f_2, \dots, f_n)$ is called as a system

of differential equations. The system where F can depend on the independent variable t is called as a non-autonomous system [2] otherwise an autonomous system which can be written as $X' = F(X)$. The system $X' = F(X,t)$ is said to be linear or non-linear according as the function F is linear or non-linear. Every solution of the system represents a curve in R^n which is called a *trajectory*.[3] Trajectories help us to study the qualitative behavior of a system. The function F is also called as a *vector field* [4]. The vector field always dictates the velocity vector X' for each X. A picture which shows all qualitatively different trajectories of the system is called as a *phase portrait*. [4]

A linear system of differential equations can be expressed as $X' = A.X$, where A is an $n \times n$ matrix. A theorem concerning the uniqueness of the solution of a linear system is stated as follows.

1.2 Theorem (The Fundamental Existence - Uniqueness Theorem) [6]

Let E be an open subset of R^ncontaining X_0 and assume that $F \in C^1(E)$. Then there exists an $a > 0$ such that the initial value problem $X' = F(X),\ X(0) = X_0$ has a unique solution $X(t)$ on the interval $[-a,\ a]$.

The fundamental existence and uniqueness theorem can be also stated as follows. A function $F: R^n \to R^n$ is said to satisfy Lipschitz condition on a domain $D \subset R^n$ if there exists a constant α such that

$$||F(X_1 - X_2)|| \leq \alpha ||X_1 - X_2|| \text{ for all } X_1, X_2 \in D$$

If F is continuously Lipschitz then the autonomous system $X' = F(X)$ has a unique solution for an initial point $X(0) = X_0$ in the domain D.

A *critical point* [10] (equilibrium point, fixed point, stationary point) X_0 is a point that satisfies the equation $X' = F(X) = 0$. If a solution starts at this point, it remains there forever.

A critical point X_0 is called *stable* critical point of the differential equation $X' = F(X)$ if given $\epsilon > 0$, there is a $\delta > 0$, such that for all $t \geq t_0$, $||\, X(t) - X_0(t) < \epsilon$ whenever $||\, X(t_0) - X_0\,(t_0)\, || < \delta$, where $X(t)$ is a solution of $X' = F(X)$. A critical point that is not stable is called an *unstable* critical point.

2. THREE DIMENSTIONAL SYSTEMS [19]

A three-dimensional linear autonomous system has the form $X' = AX$

where

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, X' = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{bmatrix},\ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Here the coefficients a_{ij} are constants. It is clear that this system has at least one critical point $O = (0, 0, 0)$. The stable and unstable *manifolds*[5] of the critical point O are respectively defined by $E_S(O) = \{X: \Lambda^+(X) = O\}$ and $E_U(O) = \{X: \Lambda^-(X) = O\}$ where $\Lambda^+(X)$ and $\Lambda^-(X)$ are the *positive limit set* and *negative limit set* of the point X respectively.

2.1 Some Examples of Three Dimensional Linear Systems

We consider some examples of the three dimensional linear systems and discuss the nature of their fixed points and analyze the nature of the solutions.

Example 1. Consider the linear autonomous system of equations given by

$$x' = x + 2z, \quad y' = y - 3z, \quad z' = 2y + z.$$

Here, the system can be expressed as the matrix equation

$$X' = AX, \quad \text{where} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, X' = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{bmatrix},\ A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 2 & 1 \end{bmatrix}$$

Solving the equations

$$x' = 0, \quad y' = 0, \quad z' = 0,$$

we get the critical point $O = (0, 0, 0)$. The three solution curves are given by the equations

$$x(t) = \frac{-e^t}{3}\left[-3C_1 + \sqrt{6}C_2 \cos\sqrt{6}\,t - \sqrt{6}C_3 \sin\sqrt{6}\,t\right],$$

$$y(t) = \frac{\sqrt{6}e^t}{2}\left[C_2 \cos\sqrt{6}\,t - C_3 \sin\sqrt{6}\,t\right],$$

$$z(t) = e^t\left[-3C_1 + \sqrt{6}C_2 \cos\sqrt{6}\,t - \sqrt{6}C_3 \sin\sqrt{6}\,t\right],$$

where C_1, C_2, C_3 are arbitrary constants.

It can be easily verified that the characteristic polynomial of the matrix A is given by

$f(x) = x^3 - 3x^2 + 9x - 7$.

Solving $f(x) = 0$, we get the eigenvalues and the corresponding eigenvectors given by $\lambda_1 = 1,\ v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \lambda_2 = 1 + \sqrt{6}i,\ v_2 = \begin{bmatrix} -\frac{\sqrt{6}i}{3} \\ \frac{\sqrt{6}i}{2} \\ 0 \end{bmatrix},\ \lambda_3 = 1 - \sqrt{6}i,\ v_3 = \begin{bmatrix} \frac{\sqrt{6}i}{3} \\ -\frac{\sqrt{6}i}{2} \\ 0 \end{bmatrix}$

Since the real part of the imaginary eigenvalues is positive, the fixed point $O = (0, 0, 0)$ is an unstable fixed point and the curve is a rotating curve.

3. THE ROSSLER SYSTEM AND ITS SOLUTION CURVES

In this section, we will consider the three dimensional non-linear system known as the Rossler system which is given by

$$x' = -y - z,\ \ y' = x + \alpha y,\ \ z' = \beta + xz - \gamma z$$

where α, β and γ are parameters. We will investigate the trajectories of the system by fixing $\alpha = \beta = 0.2$ and varying the parameter γ. But first, we will have a brief discussion about the *limit cycle* [18] of a system. Limit cycles are the closed trajectories which are isolated i. e. the neighboring trajectories are not closed. A limit cycle is called as stable or attracting of the neighboring trajectories tend towards the limit cycle, otherwise the limit cycle is known as unstable. For two dimensional systems, Poincare - Bendixson theorem [10] states that if a trajectory lies within a closed bounded region in the plane containing no fixed points, then the trajectory approaches to a limit cycle and no strange behavior is observed in the system. However, in three dimensional or higher dimensional systems, the Poincare - Bendixson theorem is not applicable and the trajectories may be trapped and just keep moving inside a bounded region without approaching towards a fixed point or a limit cycle.

3.1 Nature of Solution Curves for Different Values of the Parameter γ

As mentioned earlier, in order to analyze the nature of the solution curves, we will keep the values of the parameters $\alpha = \beta = 0.2$ fixed and change the parameter γ and study the nature of solution curves. First, we choose $\gamma = 1$. In this case, we can observe a limit cycle of period one. The projections of the trajectories showing limit cycles of period one on the *xy*-plane, *xz*-plane and *yz*-plane and the periodic behavior of the solution curves obtained using the MATLAB programming are as shown in the figure 1.

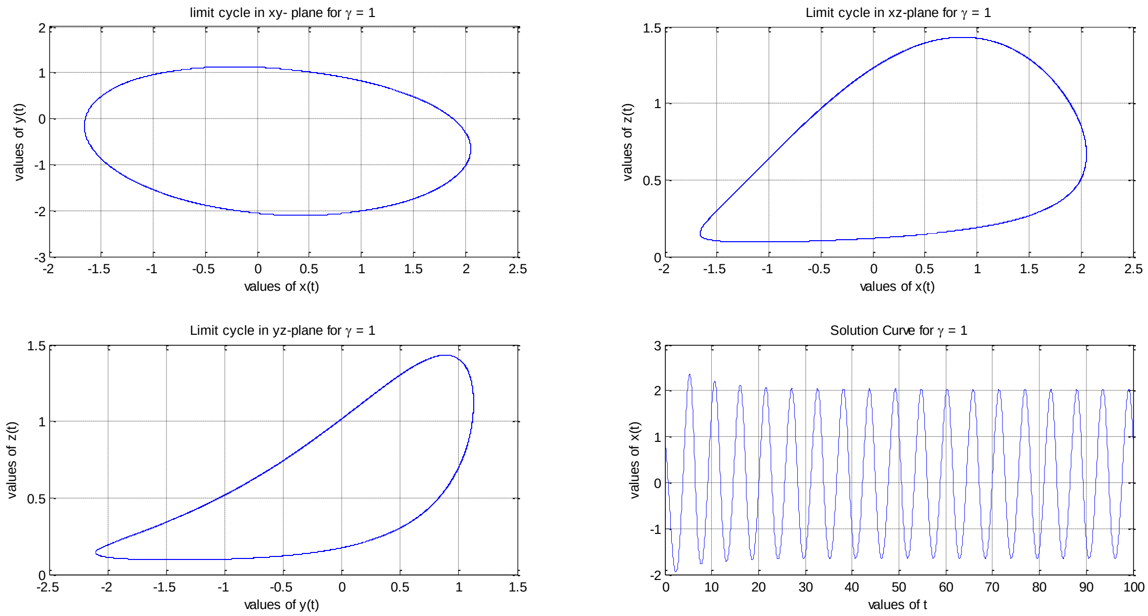

Fig. 1: Limit cycles of period one and solution curve

3.2 Period Doubling Cascade and Bifurcation Diagrams

When γ takes the value 3.4, the period one limit cycle is transformed into a limit cycle of period two. Hence in between $\gamma = 1$ and $\gamma = 3.4$, a period doubling bifurcation of limit cycles must have occurred. Such kind of bifurcations can happen only in three or higher dimensional spaces. The phase space showing the limit cycles in two dimensional spaces and the periodic solution are as shown in the figure 2.

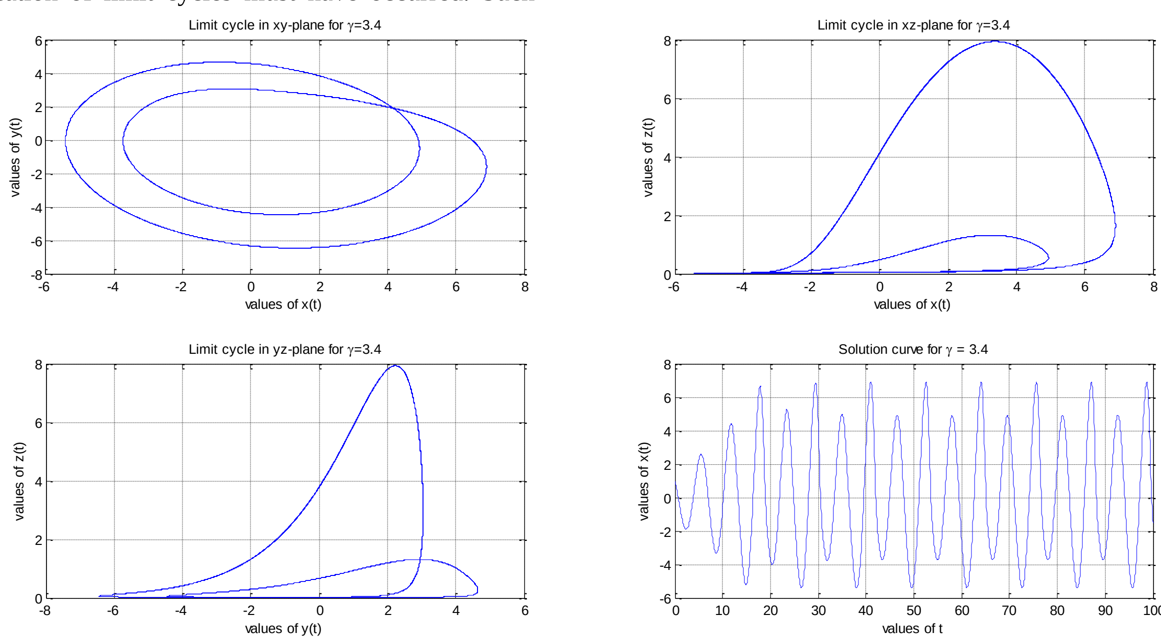

Fig. 2: Limit cycles of period two and solution curve

In the fourth subfigure of the figure 2, we can observe two distinct amplitudes. For the value $\gamma = 4$, a period four limit cycle is seen. This kind of period doubling phenomenon is observed infinitely many times as we go on increasing the values of the parameter γ. However, there is no exact method to find out the accurate values of the parameter where these bifurcations take place. The bifurcation diagram for the three parameters keeping any two of them constants are constructed using MATLAB and are as shown in figures 3, 4 and 5. The figure 5 showing the bifurcation diagram for the parameter γ can give us a clue about the approximate values of the parameter where bifurcations happen. It can be observed that the bifurcation diagrams are analogous the bifurcation diagram of the logistic family as shown in the figure 6.

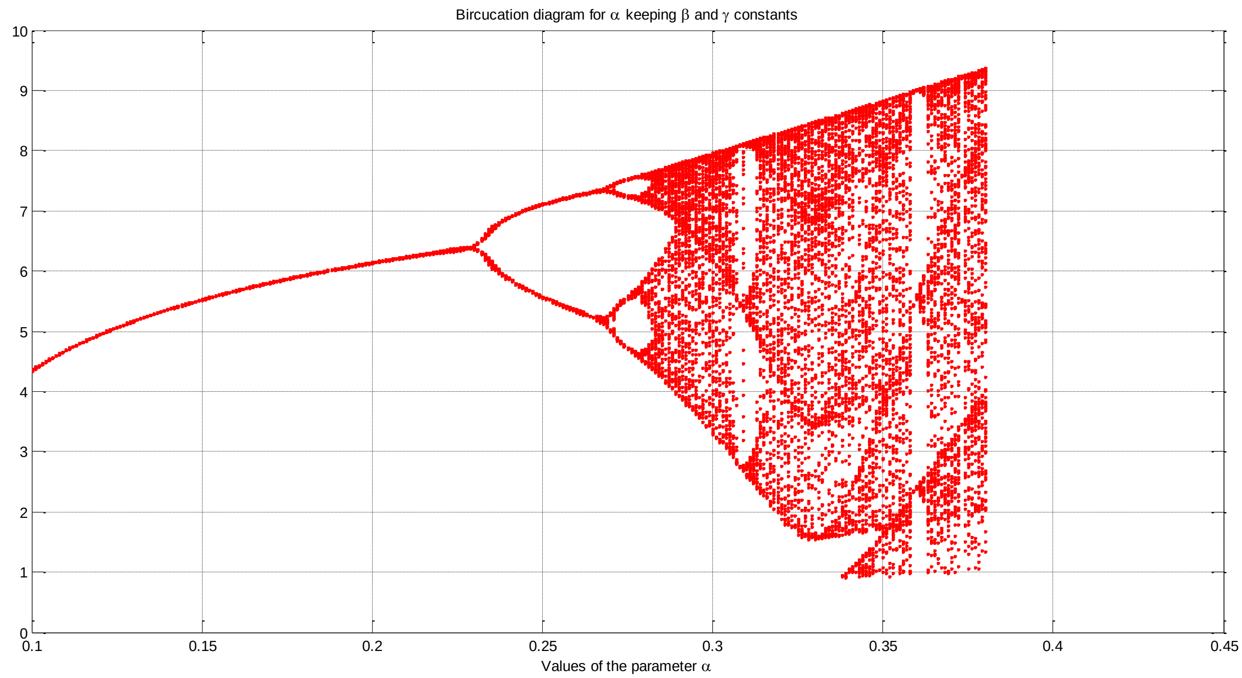

Fig. 3: Bifurcation diagram for α

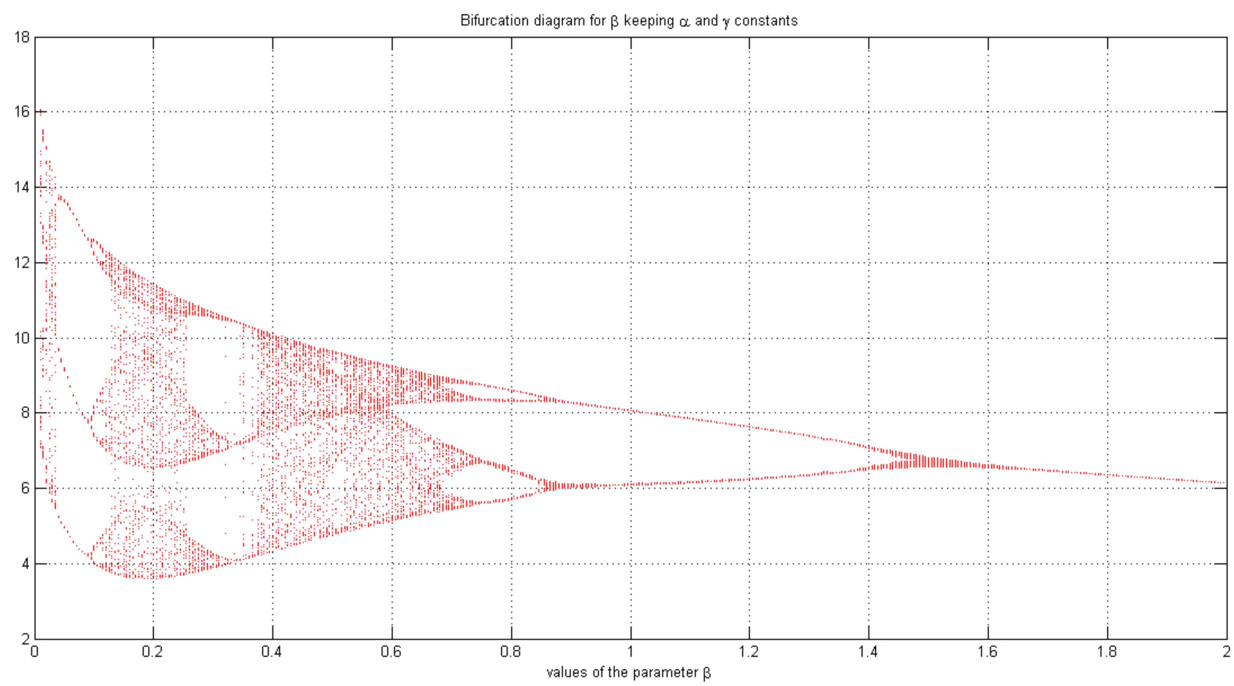

Fig. 4: Bifurcation diagram for β

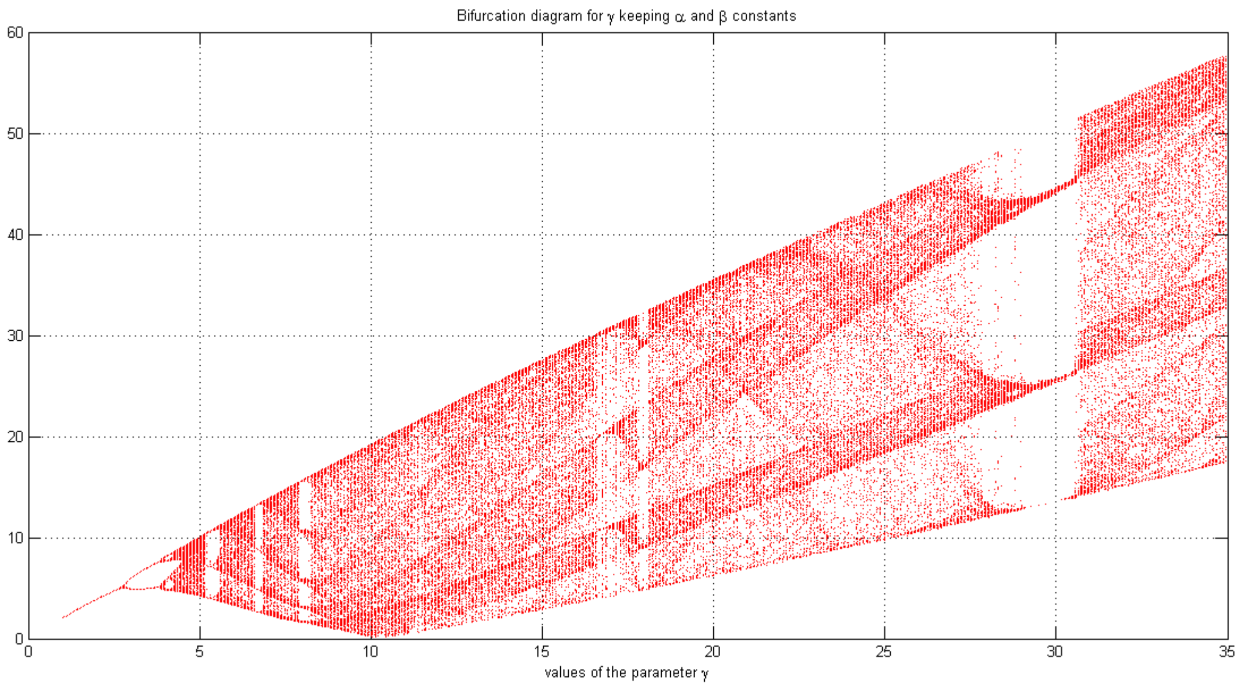

Fig. 5: Bifurcation diagram for γ

Fig. 6: Bifurcation diagram for the logistic family

3.3 Strange Atrractor

When the parameter γ takes the value 6.3, a strange attractor is observed. The strange attractor is as shown in figure 7. Observing the strange attractor, we notice that the solution curves are spirled in the *xy*-plane and then they escape out in the third dimension without intersecting with the curves in the *xy*-plane. The phase plane of the limit cycles in the three dimensional space for different values of the parameter γ can be ibserved in figure 7.

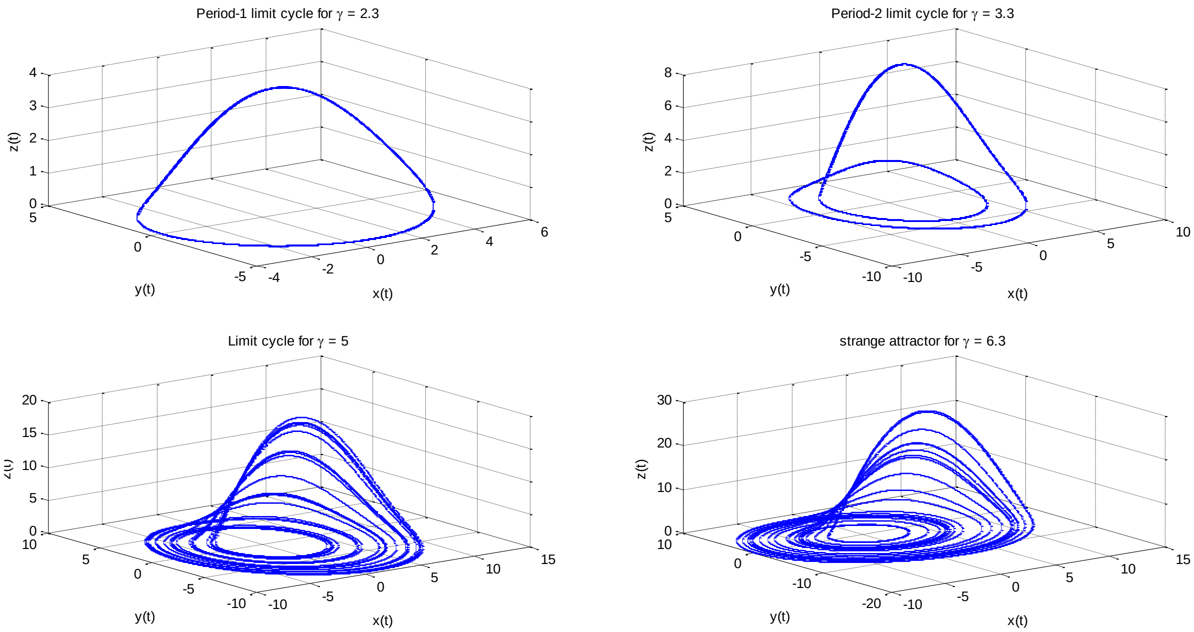

Fig. 7: limit cycles in three dimensional space

The presence of the strange attractor is one of the features of the chaotic systems. The strange attractor shows Holf bifurcation from stationary to periodic motion and a series of period doubling bifurcations leading to chaos. It can be observed that the reinjection of the trajectories is faster than the spiraling out of the trajectories fulfilling the Shilnikov criterion of chaos. The projection of the strange attractor in two dimensional spaces and the non periodic nature of the solution curves is as shown in figure 8.

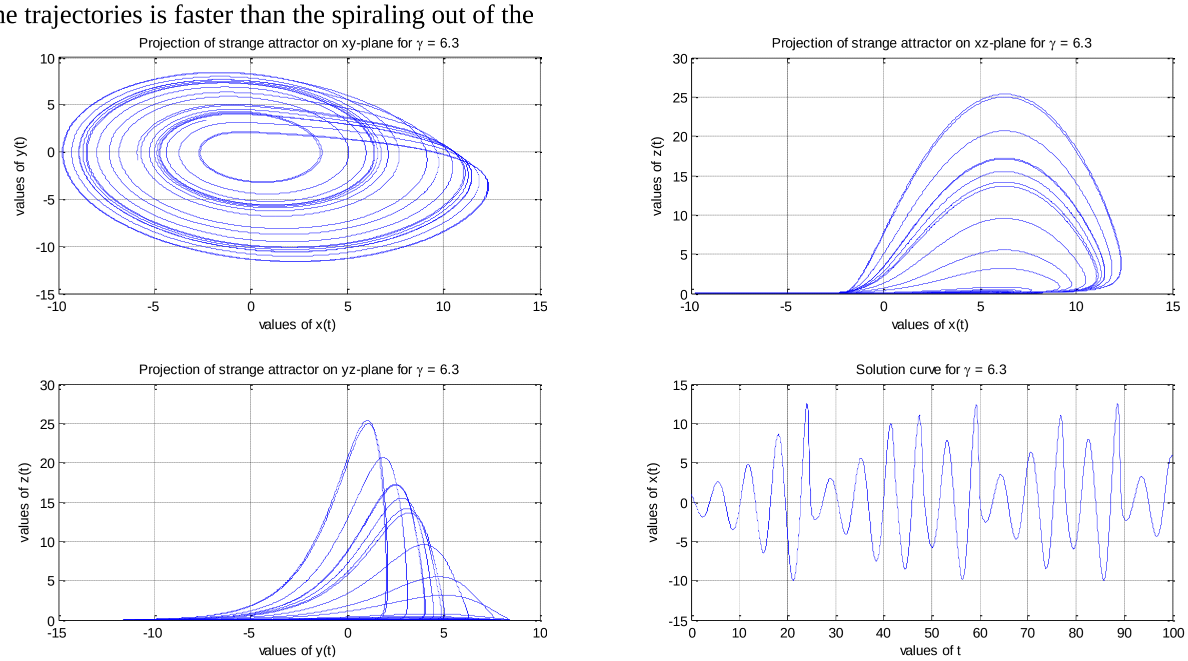

Fig. 8: Projection of strange attractor in 2-dimensional spaces and a solution curve

A strange attractor can also be found in a two dimensional quadratic mapping the Henon map $f: R^2 \rightarrow R^2$ which is defined by $f(x,\ y) = (1 - \alpha x^2 + y,\ \beta x)$, where α and β are parameters. By Kulkarni P R [15], it has been proved that keeping $\beta = 0.4$ fixed and varying the other parameter α, a period 1-cycle is observed for $\alpha = 0.2$, a period 2-cycle is observed for $\alpha = 0.5$, a period 4-cycle is observed for $\alpha = 0.9$ and so on. As α becomes greater than 1.01, this period doubling behavior is not observed, showing thereby non-periodic limit cycles. For $\alpha = 1.2$, a strange attractor is observed.

3.4 Poincare Sections

When we slice a strange attractor by a plane, the resulting two dimensional picture is known as the Poincare section. The Poincare section of the strange attractor for the parameter value $\gamma = 6.3$ is as shown in figure 9. In the Poincare section of the strange attractor, we can observe an infinite set consisting of points representing trajectories that escape out in the third dimension with certain gaps in between them. This kind of section has a fractal dimension and it is a Cantor set.

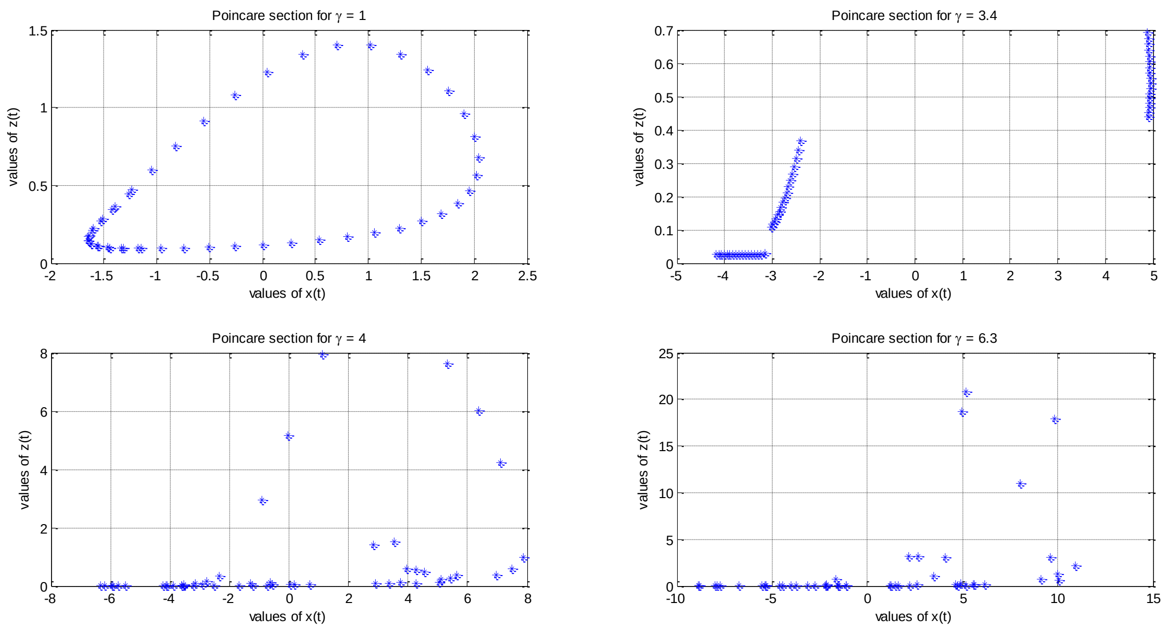

Fig. 9: Poincare sections

3.5 Sensitivity to Close Initial Conditions

Sensitive dependence on initial conditions means that on a given set of initial conditions which vary with neglisible quantities, the trajectories or motions of the system shows a totally different behavior as the time passes. Figure 10 shows that with two sets of initial conditions $x(0) = 1, y(0) = 1, z(0) = 0$ and $x(0) = 0.9, y(0) = 1, z(0) = 0$, the two trajectories though show similar pattern for a short time period, but as the time goes on increasing, they diverge rapidly and after a long period of time, they have different features. In figure 10, we can observe that there are different kind of oscillations and humps as the time t passes the value 110. This sensitive dependence on initial conditions indicate that the long term prediction of the neraby trajectories is practically impossible in such systems where the small differences in the initial conditions are amplified on a large scale as the system evolves over time.

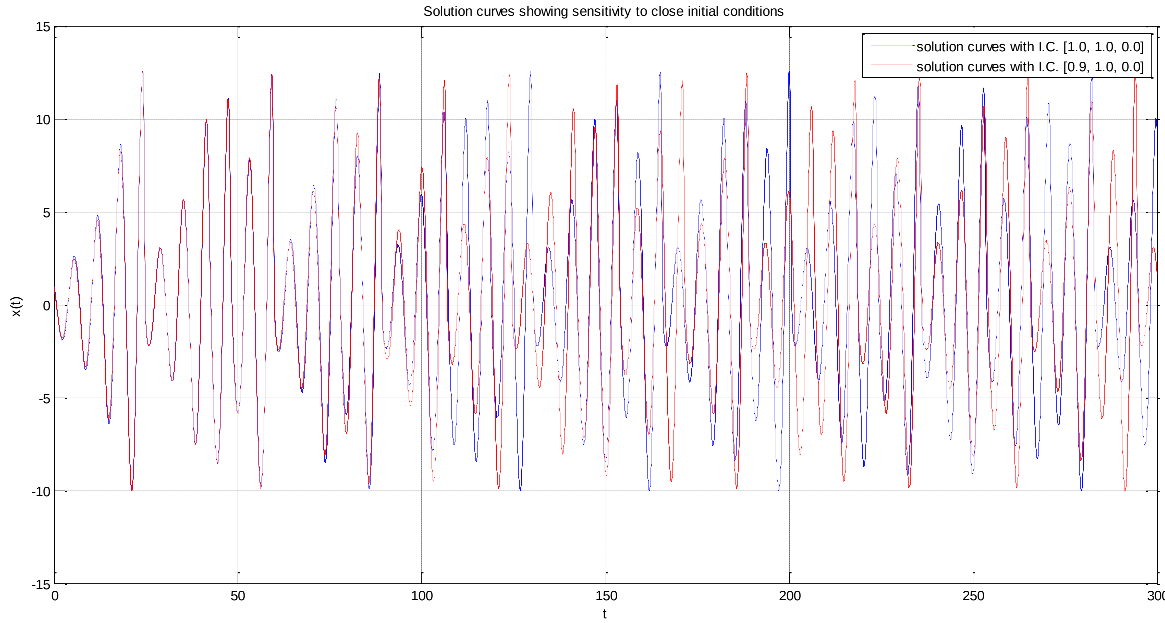

Fig. 10: Sensitive dependence on initial conditions

4. CONCLUSIONS

Most of the phenomenon occurring in natural systems, non-linear motions, mechanical vibrations [13, 16] etc. exhibit a very strange characteristic known as *chaos*. Although there is no worldwide accepted definition of chaos among scientists, there are certain common features that are shown by chaotic systems and which the most of the scientists agree upon. Chaos is the behavior of a system where there is predictability or periodic motion for certain parameter values, but as the parameter values change or as there is a small change in the initial conditions, or as the system evolves over time, this predictable nature of the system turns out to be random or unpredictable one. Some of the common feature of a chaotic system are as follows.

1. Most of the systems which are non-linear in nature show chaotic behavior.
2. Predictability over a short period of time.
3. Sensitivity on initial conditions.
4. Unpredictable long term behavior.
5. Cascade of infinitely many times period doubling.
6. Appearance of the strange attractor.
7. Fractal dimension of the strange attractor.
8. Positive Lyapunov constant.
9. Fractal dimension of Poincare section of the strange attractor.

Most of these features are observed in case of the Rossler system and thus, we have shown that the Rossler system is chaotic in nature. Many ideas can be utilized to verify that a discrete one dimensional one parameter family of mappings is chaotic. It can be observed from the bifurcation diagram that for an approximate value of $\gamma = 5.3$, there is a period three limit cycle. A well-known paper by James Yorke and T-Y Li [7] also guarantees the chaos in the Rossler system. The author Kulkarni P. R.[11, 14] has obtained the values of the parameter c for which the family of mappings $f_c(x) = x^2 - x + c$ has a period-3 orbit and thus proved that it is chaotic. Chyi-Lung Lin and Mon-Ling Shei[1] have proved that the logistic family of mappings $f(x) = \mu x(1-x)$ is topologically conjugate to the mapping $f(x) = (2-\mu)x(1-x)$ and hence is chaotic in nature. Authors Kulkarni P. R. and Borkar V. C.[12] have obtained a topological conjugacy of the family of mappings $f_c(x) = x^2 - x + c$ with the shift mapping σ and proved the chaotic nature of $f_c(x) = x^2 - x + c$ in the sense of Devaney R. L.[8]. Controlling of chaos one of the most challenging tasks before researchers all over the world as it is too difficult to have predictions about the exact parameter values for which a system falls into a chaotic regime.

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